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Some remarks on the Lie group of point transformations for the harmonic oscillator

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Abstract. Some aspects of the symmetry group SL(3, R) of the simple harmonic oscillator are revisited. A realisation of the general element of the group in Newtonian spacetime is obtained, with the eight parameters included. The group is shown to be SL(3, R) in fact, without recourse to the Lie algebra. Finally, in this way, the group law of binary composition of the parameters is calculated easily in the group manifold.

1. Introduction

Over the recent years there has been a considerable amount of literature devoted to the study of the symmetries of classical harmonic oscillator systems, both time independent (Anderson and Davison 1974, Wulfman and Wybourne 1976, Lutzky 1978a, Gordon 1986) and time dependent (Günther and Leach 1977, Lutzky 1978b, Leach 1980a, b, Prince and Eliezer 1980; cf also Lewis 1968).

Ten years ago the full Lie group of point transformations of the simple harmonic oscillator was successfully identified as SL(3, R) (Wulfman and Wybourne 1976). The purpose of this paper is to discuss this subject further, presenting an 'enlargement' of the previous work by Wulfman and Wybourne (1976). Interesting as that work indeed is, it is far from being a complete analysis of the role played by SL(3, R) as the spacetime symmetry group of the simple harmonic oscillator, since that paper presents the spacetime realisations of the group only through its one-parameter subgroups. Clearly, the description of a Lie group by means of its one-parameter subgroups is *not* complete, even if one knows the associated Lie algebra. The knowledge of the *general form* of the realisation of the elements of the group (in terms of an admissible set of essential parameters), as well as the knowledge of the *binary composition law* of the parameters (i.e. the group law (Racah 1965)), are still missing in such an approach. To state it otherwise, such a description by means of the one-parameter transformations would be as incomplete as, for instance, a description of rotations by setting separately the one-parameter rotations about the cartesian axis. Plainly, what one requires in such cases is an *analytic continuation* of these one-parameter elements of the group over the group manifold, so that the combination of two general elements may afford the group law. In this fashion, one gets knowledge of the relevant group structure and its particular realisation within the theory.

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Certainly, in the case of *non-linear realisations* of Lie groups, as they usually appear in classical mechanics, the direct exponentiation of a general element of the Lie algebra (i.e. of a linear combination of the infinitesimal differential operators acting on the variables at hand) becomes so wild that it is usually impossible to obtain the desired integrated form, unless one is handling a rather trivial exceptional case. Therefore, the problem must be faced *ab initio* under a different perspective, and recourse to the powerful techniques of *similarity analysis* of differential equations (Bluman and Cole 1974, Ovsiannikov 1982) seems to be unavoidable in this case.

Let us remark from the outset that our calculations are entirely *coordinate dependent* (both in spacetime *and* in the parameter space of the group), since our interest in this paper is to solve the following problems: (*a*) to obtain an *explicit realisation* of the full symmetry group in the spacetime of the system, with *all* its parameters included; and (*b*) to obtain a local computation of the *group law* (as distinct from calculations of the Lie algebra), i.e. of the binary composition law of the parameters (Racah 1965), in terms of the *particular parametrisation* of the elements of the group used in the solution of the previous problem.

Accordingly, we aim to show in this paper the general form of point transformations of the simple harmonic oscillator in two-dimensional Newtonian spacetime (§ 2), i.e. we obtain *eight-parameter-dependent transformations* in the variables t and x which leave invariant the equation of motion $\ddot{x}+\omega^2x=0$. Then we easily obtain the infinitesimal operators and the Lie algebra (§ 3), which we compare briefly with the corresponding results of Wulfman and Wybourne (1976). As a byproduct of this work, in § 4 we prove explicitly and without recourse to the Lie algebra that the group of transformations obtained in § 2 corresponds to SL(3, R). In § 5 we solve the problem of obtaining the group law on the group manifold (using the parametrisation of the group as presented in § 2). Finally, in § 6 we briefly present some conclusions and perspectives of this work. (The paper ends with an appendix which supplies some details of the integration process conducive to the general solution of the problem.)

2. Point transformations of the harmonic oscillator in Newtonian spacetime

We seek those spacetime point transformations $(t, x)\rightarrow(t', x')$ that leave the equation of motion of the one-dimensional harmonic oscillator form invariant. Namely, we consider the transformation of variables

$$\begin{aligned} t' &= T(t, x) \\ x' &= S(t, x) \end{aligned} \tag{2.1}$$

with non-vanishing Jacobian

$$J(t, x) = T_tS_x - T_xS_t \neq 0 \tag{2.2}$$

and such that

$$\ddot{x}+\omega^2x=0 \Leftrightarrow \ddot{x}'+\omega^2x'=0 \tag{2.3}$$

where, clearly, $\ddot{x}'=\mathrm{d}^2x'/\mathrm{d}t'^2$. The extended transformations $\dot{x}\rightarrow\dot{x}'$ and $\ddot{x}\rightarrow\ddot{x}'$ of (2.1) are formally given by

$$\dot{x}' = (T_t+T_x\dot{x})^{-1}(S_t+S_x\dot{x}) \tag{2.4}$$

and

$$\ddot{x}' = (T_t + T_x\dot{x})^{-3}[(T_tS_{tt} - T_{tt}S_t) + (2T_tS_{tx} + T_xS_{tt} - 2T_{tx}S_t - T_{tt}S_x)\dot{x}$$
$$+ (T_tS_{xx} + 2T_xS_{tx} - T_{xx}S_t - 2T_{tx}S_x)\dot{x}^2$$
$$+ (T_xS_{xx} - T_{xx}S_x)\dot{x}^3 + (T_tS_x - T_xS_t)\ddot{x}]. \tag{2.5}$$

Now, if one substitutes from (2.5) into (2.3), and separates the coefficients of the different powers of $\dot{x}$, one obtains a set of homogeneous differential equations for T and S, i.e. one gets

$$T_xS_{xx} - T_{xx}S_x + \omega^2 T_x^3 S = 0 \tag{2.6}$$

$$T_tS_{tt} - T_{tt}S_t + \omega^2[T_t^3 S - (T_tS_x - T_xS_t)x] = 0 \tag{2.7}$$

$$T_xS_{tt} - T_{tt}S_x + 2(T_tS_{tx} - T_{tx}S_t) + 3\omega^2 T_t^2 T_x S = 0 \tag{2.8}$$

$$T_tS_{xx} - T_{xx}S_t + 2(T_xS_{tx} - T_{tx}S_x) + 3\omega^2 T_t T_x^2 S = 0. \tag{2.9}$$

The general solution of these equations affords a realisation of a Lie group having *no more* than eight essential (real) parameters (Bianchi 1928), which may be adjusted suitably by means of the integration constants of (2.6)-(2.9), and which operates as the symmetry group in the Newtonian spacetime of the system. The detailed integration of equations (2.6)-(2.9) is quite involved and we discuss it in the appendix of this paper.

A glance at equations (2.6)-(2.9) shows immediately that there are only two cases worthy of interest, depending on whether one has (case I) $T_x \neq 0$ or (case II) $T_x = 0$. This classification of the problem is not as trivial as it seems at first sight, since the very reason for this distinction is group-theoretic. In fact, when one starts solving this problem, it is not reasonable to consider *a priori* that case II is merely a 'special' instance of the 'general' case I, since, certainly, one must have $T_x = 0$ at the identity, and therefore there is no way of knowing in advance which integration scheme is the better in case I in order to avoid leaving the identity component of the transformation group out of the picture.

The previous remarks set the basis of the method we follow in this paper. We consider cases I and II separately (though here we show the detailed calculations correspnding to the first case only, for the sake of brevity). In the appendix we assume $T_x \neq 0$ and we follow a straightforward integration scheme for solving equations (2.6)-(2.9), without worrying about the possible group-theoretic meaning of the constants of integration. Afterwards, we adjust the integration constants by means of suitable 'initial conditions' which we introduce at the event $(t, x) = (0, 0)$. In this fashion we get a reasonable parametrisation of the point transformation group we are looking for. Indeed, if one solves case II separately, using the same 'initial conditions', it can be shown that case II is included as a *subgroup* of the transformation found in case I. Hence, in this manner we get an admissible analytic continuation of the solution of the problem in the group manifold.

According to the integration procedure presented in the appendix, we obtain the following solution (case I):

$$\omega T(t, x) = \tan^{-1} k_4 + \tan^{-1}(\psi_0(t)x + \psi_5(t)) \tag{2.10}$$

$$S(t, x) = \psi_1(t)\cos\omega T(t, x) + \psi_2(t)\sin\omega T(t, x) \tag{2.11}$$

where we have defined

$$\psi_0(t) = k_6 \sec(\omega t + k_1) \tag{2.12}$$

$$\psi_1(t) = \frac{k_2}{\omega}\left(\frac{k_2 k_6}{\omega(1+k_4^2)}\right)^{1/2} \tan(\omega t + k_1) + k_8 \tag{2.13}$$

$$\psi_2(t) = k_4 \psi_1(t) + k_5 \tag{2.14}$$

$$\psi_5(t) = (k_3 k_6/\omega) \tan(\omega t + k_1) + k_7. \tag{2.15}$$

The constants of integration, $k_1, \ldots, k_8$, look quite bizarre since they figure in this solution in just the same way they appeared during the integration process itself. As it stands, it is not possible to obtain the identity transformation ($T = t$, $S = x$) within this solution by means of a simple adjustment of the k. Hence, let us introduce the following set of 'initial conditions' at $(t, x) = (0, 0)$:

$$\begin{aligned} &A_1 = T(0,0) && A_2 = S(0,0) \\ &A_3 = T_t(0,0) && A_4 = S_x(0,0) \\ &A_5 = T_x(0,0) && A_6 = S_t(0,0) \\ &A_7 = \tfrac{1}{2}T_{tt}(0,0) && A_8 = \tfrac{1}{2}S_{xx}(0,0) \end{aligned} \tag{2.16}$$

such that the Jacobian (2.2) is different from zero at $(t, x) = (0, 0)$, i.e.

$$J_0 = A_3 A_4 - A_5 A_6 \neq 0 \tag{2.17}$$

(cf Aguirre and Krause 1984). In this fashion one obtains enough algebraic relations between the k and the A, and one reparametrises the solution (2.10)-(2.15) in terms of the A. We omit the details of this tedious calculation. It yields the following spacetime coordinate transformations:

$$t' = \frac{1}{\omega}\sin^{-1}\left(\frac{q^5 \sin\omega t + q^6 x}{[(q^5 \sin\omega t + q^6 x)^2 + (\cos\omega t + q^7 \sin\omega t + q^8 x)^2]^{1/2}}\right) + q^1 \tag{2.18}$$

$$x' = \frac{q^2 \cos\omega t + q^3 \sin\omega t + q^4 x}{[(q^5 \sin\omega t + q^6 x)^2 + (\cos\omega t + q^7 \sin\omega t + q^8 x)^2]^{1/2}} \tag{2.19}$$

where we have defined new parameters:

$$\begin{aligned} &q^1 = A_1 \qquad q^2 = A_2 \qquad q^3 = \frac{1}{\omega}(A_6 - A_2 A_7/A_3) \\ &q^4 = A_4 - A_2(A_8 + \tfrac{1}{2}\omega^2 A_2 A_5^2)/A_4 \qquad q^5 = A_3 \\ &q^6 = \omega A_5 \qquad q^7 = -\frac{1}{\omega}(A_7/A_3) \qquad q^8 = -(A_8 + \tfrac{1}{2}\omega^2 A_2 A_5^2)/A_4. \end{aligned} \tag{2.20}$$

Equations (2.18) and (2.19) entail our final answer. Let us remark that if one sets $q^6 = q^8 = 0$ in (2.18)-(2.20) one arrives precisely at the solution one obtains directly in case II (i.e. assuming $T_x = 0$), if one uses the same 'initial conditions' (2.16) (with $A_5 = A_8 = 0$).

Once obtained, two features can be checked in a straightforward (albeit lengthy) calculation, concerning transformations (2.18) and (2.19). Firstly, by calculating their first and second extensions (i.e. $\dot{x} \to \dot{x}'$ and $\ddot{x} \to \ddot{x}'$), one proves *directly* that equations (2.18) and (2.19) are consistent with the fundamental implication stated in equation (2.3) (as they should be). Secondly, by performing two successive coordinate transformations, $(t, x) \to^{q} (t', x') \to^{q'} (t'', x'')$, of the form (2.18) and (2.19), with parameters $q = (q^1, \ldots, q^8)$ and $q' = (q'^1, \ldots, q'^8)$, respectively, one obtains a coordinate transformation, $(t, x) \to^{q''} (t'', x'')$, of the *same form* (2.18) and (2.19), with parameters $q'' = (q''^1, \ldots, q''^8)$ that are functions of q' and q, say: $q''^a = g^a(q'; q)$, $a = 1, \ldots, 8$ (as expected). (We here omit the calculations leading to these controls. The functions $g^a(q'; q)$ shall be obtained in § 5, following a more convenient approach than the 'direct' one.)

Clearly, the identity transformation corresponds to the following choice of parameters:

$$\begin{aligned} &q^1 = q^2 = q^3 = 0 \\ &q^4 = q^5 = 1 \\ &q^6 = q^7 = q^8 = 0. \end{aligned} \tag{2.21}$$

The Jacobian of (2.18) and (2.19) is

$$J(t, x) = \psi(q)[(q^5 \sin \omega t + q^6 x)^2 + (\cos \omega t + q^7 \sin \omega t + q^8 x)^2]^{-3/2} \tag{2.22}$$

where $\psi(q)$ stands for

$$\psi(q) = q^4 q^5 - q^3 q^6 + q^2(q^6 q^7 - q^5 q^8). \tag{2.23}$$

Thus, whenever $\psi(q) \neq 0$, equations (2.18) and (2.19) entail an admissible transformation of coordinates, which holds good everywhere in spacetime. Bear in mind that the transformations (2.18) and (2.19) are forbidden only at the six-dimensional locus $\psi(q^2, \ldots, q^8) = 0$ in the eight-parameter space. *Outside this hypersurface*, the q satisfy the *anholonomic constraints*: $\psi(q) > 0$ or $\psi(q) < 0$, and the transformations are well defined with respect to the parameters, for $-\pi < \omega q^1 < \pi$, and $-\infty < q^a < +\infty$, $a = 2, \ldots, 8$.

In conclusion, equations (2.18) and (2.19) entail a spacetime realisation of an eight-parameter Lie group (which, according to the result of Wulfman and Wybourne (1976), must correspond to a realisation of SL(3, R)). Let us recall that one can parametrise the elements of a Lie group in different ways and that, in general, it is not possible to cover the whole group manifold by means of a single patch corresponding to just *one* set of continuous real parameters. Thus, we do not claim that our parametrisation is the 'best' one. It is good enough for our purposes in this paper, since it certainly covers an extended domain of the group including the identity.

3. Infinitesimal operators and the Lie algebra

According to equations (2.21) (which give us the coordinates of the identity point in the group manifold), one obtains from (2.18) and (2.19) the spacetime realisation of the monoparametric subgroups. Hence, in a neighbourhood of the identity one easily calculates Lie's infinitesimal operators. Recalling that $q^4 = 1 + \delta q^4$ and $q^5 = 1 + \delta q^5$, one obtains

$$Z_1 = \partial_t \tag{3.1}$$

$$Z_2 = \cos \omega t \partial_x \tag{3.2}$$

$$Z_3 = \sin \omega t \partial_x \tag{3.3}$$

$$Z_4 = x \partial_x \tag{3.4}$$

$$Z_5 = \frac{1}{2\omega} \sin 2\omega t \partial_t - x \sin^2 \omega t \partial_x \tag{3.5}$$

$$Z_6 = \frac{x}{\omega} \cos \omega t \partial_t - x^2 \sin \omega t \partial_x \tag{3.6}$$

$$Z_7 = -\frac{1}{\omega} \sin^2 \omega t \partial_t - \tfrac{1}{2} x \sin 2\omega t \partial_x \tag{3.7}$$

$$Z_8 = -\frac{x}{\omega} \sin \omega t \partial_t - x^2 \cos \omega t \partial_x . \tag{3.8}$$

Incidentally, it is interesting to observe that the following linear combinations of these operators:

$$\begin{aligned} &X_1 = Z_3 - Z_6 && X_2 = Z_3 + Z_6 \\ &X_3 = Z_2 - Z_8 && X_4 = Z_2 + Z_8 \\ &X_5 = Z_1 && X_6 = Z_4 \\ &X_7 = Z_4 + 2Z_5 && X_8 = \frac{1}{\omega} Z_1 + 2Z_7 \end{aligned} \tag{3.9}$$

correspond precisely to the infinitesimal operators for the harmonic oscillator, as obtained by Wulfman and Wybourne (1976). We present the Lie algebra of the Z operators in table 1. This Lie algebra is isomorphic with the Lie algebra of the X operators, which had been already discussed in detail by Wulfman and Wybourne (1976).

Table 1. The commutators $[Z_a, Z_b]$, a, $b = 1, \ldots, 8$, of the infinitesimal operators.

	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7	Z_8
Z_1	0	$-\omega Z_3$	ωZ_2	0	$Z_1 + 2\omega Z_7$	ωZ_8	$-\omega(Z_4 + 2Z_5)$	$-\omega Z_6$
Z_2	ωZ_3	0	0	Z_2	0	$\frac{1}{\omega} Z_1 + Z_7$	$-Z_3$	$-(2Z_4 + Z_5)$
Z_3	$-\omega Z_2$	0	0	Z_3	$-Z_3$	$-Z_4 + Z_5$	0	Z_7
Z_4	0	$-Z_2$	$-Z_3$	0	0	Z_6	0	Z_8
Z_5	$-(Z_1 + 2\omega Z_7)$	0	Z_3	0	0	$-Z_6$	Z_7	0
Z_6	$-\omega Z_8$	$-\left(\frac{1}{\omega} Z_1 + Z_7\right)$	$Z_4 - Z_5$	$-Z_6$	Z_6	0	Z_8	0
Z_7	$\omega(Z_4 + 2Z_5)$	Z_3	0	0	$-Z_7$	$-Z_8$	0	0
Z_8	Z_6	$2Z_4 + Z_5$	$-Z_7$	$-Z_8$	0	0	0	0

4. SL(3, *R*)

We are in a position to discuss briefly the issue of SL(3, R) (Wulfman and Wybourne 1976), since equations (2.18) and (2.19) afford a realisation of the group which holds everywhere in spacetime $\{(t, x)\}$. To this end, we better handle the transformations (2.18) and (2.19) in a more 'compact' fashion. Thus, let us write

$$\begin{bmatrix} \cos \omega t' \\ \sin \omega t' \\ x' \end{bmatrix} = \phi(\cos \omega t, \sin \omega t, x; q) \begin{bmatrix} \cos \omega q^1 & -\sin \omega q^1 & 0 \\ \sin \omega q^1 & \cos \omega q^1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & q^7 & q^8 \\ 0 & q^5 & q^6 \\ q^2 & q^3 & q^4 \end{bmatrix} \begin{bmatrix} \cos \omega t \\ \sin \omega t \\ x \end{bmatrix} \tag{4.1}$$

where

$$\phi(\cos \omega t, \sin \omega t, x; q) = [(q^5 \sin \omega t + q^6 x)^2 + (\cos \omega t + q^7 \sin \omega t + q^8 x)^2]^{-1/2}. \tag{4.2}$$

This transformation is *equivalent* to the transformation scheme stated in equations (2.18) and (2.19).

Let us then rewrite equation (4.1), symbolically, as

$$\boldsymbol{v}' = \phi(\boldsymbol{v}; q) \boldsymbol{R}(q) \cdot \boldsymbol{Q}(q) \cdot \boldsymbol{v} \tag{4.3}$$

the meaning of which is clear, i.e. $\boldsymbol{v} = (\cos \omega t, \sin \omega t, x)$ (transposed). The matrices $\boldsymbol{R} \cdot \boldsymbol{Q}$ are non-singular; indeed, one has

$$\det(\boldsymbol{R} \cdot \boldsymbol{Q}) = \det(\boldsymbol{Q}) = \psi(q) = q^4 q^5 - q^3 q^6 + q^2(q^6 q^7 - q^5 q^8) \neq 0. \tag{4.4}$$

Furthermore, the Jacobian $J(t, x; q)$ of (2.18) and (2.19) is given by (cf equations (2.22) and (2.23))

$$J(t, x; q) = (\phi(\boldsymbol{v}; q))^3 \det(\boldsymbol{Q}(q)). \tag{4.5}$$

Then, as was already remarked in § 2, upon performing two successive transformations, one has

$$\boldsymbol{v}'' = \phi' \phi \boldsymbol{R}' \cdot \boldsymbol{Q}' \cdot \boldsymbol{R} \cdot \boldsymbol{Q} \cdot \boldsymbol{v} = \phi'' \boldsymbol{R}'' \cdot \boldsymbol{Q}'' \cdot \boldsymbol{v} \tag{4.6}$$

for *every* point $\boldsymbol{v} = (\cos \omega t, \sin \omega t, x)$. Of course, $\boldsymbol{R}'$ stands for $\boldsymbol{R}(q')$, etc, so that ϕ and ϕ'' depend directly on $\boldsymbol{v}$, according to (4.2), while ϕ' depends on $\boldsymbol{v}$ through (4.3) *and* (4.2). However, because of the well known law of multiplication of the Jacobians, $J'' = J'J$, according to (4.5) we get

$$\phi''^3 \det(\boldsymbol{Q}'') = (\phi' \phi)^3 \det(\boldsymbol{Q}') \det(\boldsymbol{Q}). \tag{4.7}$$

Thus we see that the combination (as yet unknown)

$$\lambda(q'; q) = \frac{\phi' \phi}{\phi''} = \left(\frac{\det(\boldsymbol{Q}'')}{\det(\boldsymbol{Q}' \cdot \boldsymbol{Q})} \right)^{1/3} = \left(\frac{\psi(g(q'; q))}{\psi(q')\psi(q)} \right)^{1/3} \tag{4.8}$$

is a function of the parameters q' and q, but is *independent* of $\boldsymbol{v}$. (Let us observe that the parameters q'', which figure in $\boldsymbol{Q}''$, are functions of the parameters q' and q, which figure in $\boldsymbol{Q}'$ and $\boldsymbol{Q}$ respectively, because of the group law, i.e. $q'' = g(q'; q)$, cf § 5.) Finally then, the important point to note is that from (4.6) and (4.8) we obtain

$$\frac{\boldsymbol{R}(q') \cdot \boldsymbol{Q}(q') \cdot \boldsymbol{R}(q) \cdot \boldsymbol{Q}(q) \cdot \boldsymbol{v}}{(\psi(q')\psi(q))^{1/3}} = \frac{\boldsymbol{R}(q'') \cdot \boldsymbol{Q}(q'') \cdot \boldsymbol{v}}{(\psi(q''))^{1/3}} \tag{4.9}$$

which holds for *all* $\boldsymbol{v}$, i.e. one has

$$\frac{\boldsymbol{R}(q')\cdot\boldsymbol{Q}(q')}{(\psi(q'))^{1/3}}\cdot\frac{\boldsymbol{R}(q)\cdot\boldsymbol{Q}(q)}{(\psi(q))^{1/3}}=\frac{\boldsymbol{R}(q'')\cdot\boldsymbol{Q}(q'')}{(\psi(q''))^{1/3}}\in \mathrm{SL}(3,R) \tag{4.10}$$

since obviously (cf (4.4))

$$\det\left[\frac{\boldsymbol{R}(q)\cdot\boldsymbol{Q}(q)}{(\psi(q))^{1/3}}\right]=1. \tag{4.11}$$

Hence, in the present realisation of the group, the rule of combination of two successive transformations of coordinates corresponds precisely to the rule of multiplication of two unimodular 3×3 real matrices. This simple result exhibits explicitly the fact that the elements of the symmetry group of the one-dimensional harmonic oscillator, as realised in (2.18) and (2.19), correspond *necessarily* to elements of the group SL(3, R) (Wulfman and Wybourne 1976).

5. The group law

We finally present the group law, i.e. the binary combination rules of the parameters $q^1,\ldots,q^8$ which figure in the transformation of variables (2.18) and (2.19) or, better, in equation (4.1). Clearly, according to the previous discussion, all one has to do in order to find the group multiplication functions (Racah 1965)

$$q''^a=g^a(q';q) \qquad a=1,\ldots,8 \tag{5.1}$$

(say) is to perform the operations indicated in (4.9) in an explicit manner. Thus we set

$$\boldsymbol{R}''\cdot\boldsymbol{Q}''=\lambda(q';q)(\boldsymbol{R}'\cdot\boldsymbol{Q}')\cdot(\boldsymbol{R}\cdot\boldsymbol{Q}) \tag{5.2}$$

where the matrices $\boldsymbol{R}\cdot\boldsymbol{Q}$ are of the form (cf (4.1))

$$\boldsymbol{R}\cdot\boldsymbol{Q}=\begin{bmatrix}\cos\omega q^1 & q^7\cos\omega q^1-q^5\sin\omega q^1 & q^8\cos\omega q^1-q^6\sin\omega q^1\\ \sin\omega q^1 & q^7\sin\omega q^1+q^5\cos\omega q^1 & q^8\sin\omega q^1+q^6\cos\omega q^1\\ q^2 & q^3 & q^4\end{bmatrix}. \tag{5.3}$$

(Recall that $\det(\boldsymbol{R}\cdot\boldsymbol{Q})=\psi(q)$.) Hence, substituting expressions (4.5) and (5.3) into (4.8) and (5.2), after some rather lengthy but obvious manipulations, one arrives at the following results:

$$q''^1=q'^1+\frac{1}{\omega}\tan^{-1}\left(\frac{q'^5\sin\omega q^1+q'^6q^2}{\cos\omega q^1+q'^7\sin\omega q^1+q'^8q^2}\right) \tag{5.4}$$

$$q''^2=\lambda(q';q)(q'^2\cos\omega q^1+q'^3\sin\omega q^1+q'^4q^2) \tag{5.5}$$

$$q''^3=\lambda(q';q)[(q'^2q^7+q'^3q^5)\cos\omega q^1+(q'^3q^7-q'^2q^5)\sin\omega q^1+q'^4q^3] \tag{5.6}$$

$$q''^4=\lambda(q';q)[(q'^2q^8+q'^3q^6)\cos\omega q^1+(q'^3q^8-q'^2q^6)\sin\omega q^1+q'^4q^4] \tag{5.7}$$

$$\begin{aligned}q''^5=\lambda^2(q';q)\{&[q'^6(q^3-q^2q^7)+(q'^5q'^8-q'^6q'^7)q^2q^5]\cos\omega q^1\\&+[q'^6q^2q^5-(q'^5q'^8-q'^6q'^7)(q^3-q^2q^7)]\sin\omega q^1+q'^5q^5\}\end{aligned} \tag{5.8}$$

$$\begin{aligned}q''^6=\lambda^2(q';q)\{&[q'^6(q^4-q^2q^8)+(q'^5q'^8-q'^6q'^7)q^2q^6]\cos\omega q^1\\&+[q'^6q^2q^6-(q'^5q'^8-q'^6q'^7)(q^4-q^2q^8)]\sin\omega q^1+q'^5q^6\}\end{aligned} \tag{5.9}$$

$$q''^7 = \lambda^2(q';q)\{[(q'^7q^5+q^7)\cos\omega q^1+(q'^7q^7-q^5)\sin\omega q^1+q'^8q^3] \times(\cos\omega q^1+q'^7\sin\omega q^1+q'^8q^2) +(q'^5q^5\cos\omega q^1+q'^5q^7\sin\omega q^1+q'^6q^3)(q'^5\sin\omega q^1+q'^6q^2)\} \quad (5.10)$$

$$q''^8 = \lambda^2(q';q)\{[(q'^7q^6+q^8)\cos\omega q^1+(q'^7q^8-q^6)\sin\omega q^1+q'^8q^4] \times(\cos\omega q^1+q'^7\sin\omega q^1+q'^8q^2) +(q'^5q^6\cos\omega q^1+q'^5q^8\sin\omega q^1+q'^6q^4)(q'^5\sin\omega q^1+q'^6q^2)\} \quad (5.11)$$

where λ is given by

$$\lambda(q';q) = [(q'^5\sin\omega q^1+q'^6q^2)^2+(\cos\omega q^1+q'^7\sin\omega q^1+q'^8q^2)^2]^{-1/2}. \quad (5.12)$$

The importance of the group multiplication functions for a given set of essential (real) parameters of a Lie group is well known (Racah 1965). Bear in mind that

$$\psi(q')\psi(q)\neq 0 \Rightarrow \psi(g(q';q))\neq 0 \quad (5.13)$$

and therefore, according to (4.8) one has, necessarily,

$$\psi(q')\psi(q)\neq 0 \Rightarrow \lambda(q';q)\neq 0. \quad (5.14)$$

Hence, the group law is well defined everywhere outside the locus $\psi(q)=0$. Nevertheless, let us emphasise that this group law has only a *local character* (distinct, however, from that of the Lie algebra).

6. Conclusions and perspectives

This paper is a report of work in progress concerning the Lie symmetries of the time-independent one-dimensional harmonic oscillator. Our interest in this subject stems from its possible physical applications in non-Abelian quantum kinematics (Krause 1985). Indeed, in a recent paper, the simple harmonic oscillator has been successfully quantised by means of the regular representation of its *Newtonian subgroup* of spacetime symmetries (Krause 1986), which, of course, is a subgroup of the full symmetry group considered in the present work. Hence, the kinematic quantisation (Krause 1985) of the full symmetry group of the harmonic oscillator seems to be an endeavour worthy of further research. We mention this motivation here, though in this paper we have not touched upon the issue of quantum kinematics.

From the standpoint of quantum kinematics, the main results of this paper have been: (1) to derive formulae for $q''^a = g^a(q';q)$, in the case of a particular parametrisation of SL(3, R), since this knowledge is required for building the *quantum kinematic model* underlying this group (Krause 1985); and (2) to derive the general formulae for $t' = T(t, x; q)$ and $x' = S(t, x; q)$ in the case of the simple harmonic oscillator, since this knowledge allows one to obtain the *wave mechanics* of the system over the spacetime arena (Krause 1986), taking exclusively into account *all* its symmetries. The problem set by SL(3, R) quantum kinematics will be tackled elsewhere.

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Appendix

Here we supply the details of the integration process conducive to the general solution of equations (2.6)-(2.9) when $T_x \neq 0$ (i.e. case I). One easily integrates (2.6) twice with respect to x, and obtains the following general form for S:

$$S(t, x) = \psi_1(t) \cos \omega T(t, x) + \psi_2(t) \sin \omega T(t, x) \tag{A1}$$

where ψ_1 and ψ_2 are arbitrary functions of t. We next substitute from (A1) into the remaining equations (2.7)-(2.9). It will be useful to define

$$\sigma(t, x) = \dot{\psi}_1 \cos \omega T + \dot{\psi}_2 \sin \omega T. \tag{A2}$$

In this way, (2.7)-(2.9) can be cast in the forms:

$$\sigma T_{xx} - 2\sigma_x T_x = 0 \tag{A3}$$

$$2\sigma T_{tx} - 3\sigma_x T_t - \sigma_t T_x = 0 \tag{A4}$$

$$\sigma T_x T_{tt} - \sigma_x T_t^2 - \sigma_t T_t T_x - \omega^2 x \sigma T_x^2 = 0 \tag{A5}$$

respectively. The formal integration of (A3) is immediate. One obtains

$$\sigma^2 = \psi_3(t) T_x \tag{A6}$$

and, therefore, equations (A4) and (A5) become

$$T_x T_{tx} - T_t T_{xx} - \tfrac{1}{3}(\dot{\psi}_3/\psi_3) T_x^2 = 0 \tag{A7}$$

$$2T_x^2 T_{tt} - T_t^2 T_{xx} - T_t T_x T_{tx} - (\dot{\psi}_3/\psi_3) T_t T_x^2 - 2\omega^2 x T_x^3 = 0. \tag{A8}$$

From (A7), one has

$$T_t / T_x = \tfrac{1}{3}(\dot{\psi}_3/\psi_3) x + \psi_4(t). \tag{A9}$$

Thus, (A8) can be separated into the following system of ordinary differential equations:

$$\dot{u} - u^2 - \omega^2 = 0 \tag{A10}$$

$$\dot{\psi}_4 - u\psi_4 = 0 \tag{A11}$$

where $u = \dot{\psi}_3/3\psi_3$. Clearly, ψ_3 and ψ_4 denote two new unknown functions of t. Since (A10) yields $u = \omega \tan(\omega t + k_1)$, one easily arrives at

$$\psi_3(t) = k_2^3 \sec^3(\omega t + k_1) \tag{A12}$$

$$\psi_4(t) = k_3 \sec(\omega t + k_1). \tag{A13}$$

Accordingly, equations (A6) and (A9) correspond to

$$\sigma^2 = k_2^3 \sec^3(\omega t + k_1) T_x \tag{A14}$$

and

$$T_t = [\omega x \tan(\omega t + k_1) + k_3 \sec(\omega t + k_1)] T_x \tag{A15}$$

respectively, where k_1, k_2, k_3 are constants of integration. Furthermore, since

$$\left(\frac{\dot{\psi}_1 \sin \omega T - \dot{\psi}_2 \cos \omega T}{\dot{\psi}_1 \cos \omega T + \dot{\psi}_2 \sin \omega T}\right)_x = \frac{\omega(\dot{\psi}_1^2 + \dot{\psi}_2^2) T_x}{(\dot{\psi}_1 \cos \omega T + \dot{\psi}_2 \sin \omega T)^2} \tag{A16}$$

from equations (A2) and (A14) (after performing one partial integration in x and using (A16)), one obtains

$$\tan \omega T = \frac{(\dot{\psi}_2/\dot{\psi}_1) + \psi_0(t)x + \psi_5(t)}{1 - (\dot{\psi}_2/\dot{\psi}_1)(\psi_0(t)x + \psi_5(t))} \tag{A17}$$

where $\psi_0(t)$ stands for

$$\psi_0(t) = \frac{\omega}{k_2^3}(\dot{\psi}_1^2 + \dot{\psi}_2^2)\cos^3(\omega t + k_1) \tag{A18}$$

and where $\psi_5(t)$ is a new function of t. Inspection of (A17) leads us to define

$$\tan \Phi = \dot{\psi}_2/\dot{\psi}_1 \tag{A19}$$

$$\tan \Psi = \psi_0 x + \psi_5 \tag{A20}$$

so that $T(t, x)$ becomes

$$\omega T = \Phi(t) + \Psi(t, x). \tag{A21}$$

Hence

$$\omega T_t = \cos^2 \Phi \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\psi}_2/\dot{\psi}_1) + (\dot{\psi}_0 x + \dot{\psi}_5)\cos^2 \Psi \tag{A22}$$

$$\omega T_x = \psi_0 \cos^2 \Psi \tag{A23}$$

follow. If one substitutes these expressions for T_t and T_x into (A15), recalling (A19) and (A20), and separates the coefficients for different powers of x, after integrating in t, one finds

$$\psi_2 = k_4\psi_1 + k_5 \tag{A24}$$

$$\psi_0 = k_6 \sec(\omega t + k_1) \tag{A25}$$

$$\psi_5 = \frac{1}{\omega} k_3 k_6 \tan(\omega t + k_1) + k_7 \tag{A26}$$

where k_4, k_5, k_6 and k_7 are new constants of integration. Finally, if one substitutes from (A24) and (A25) into (A18) and performs a last integration in t, one obtains

$$\psi_1 = \frac{k_2}{\omega}\left(\frac{k_2 k_6}{\omega(1 + k_4^2)}\right)^{1/2} \tan(\omega t + k_1) + k_8. \tag{A27}$$

In this manner, the rather lengthy (albeit straightforward) process of integration is complete. From (A19) and (A24) we see that Φ is a constant, i.e.

$$\tan \Phi = k_4. \tag{A28}$$

All that remains to be done is to recollect our results: substitute from (A25) and (A26) into (A20), and from (A20) and (A28) into (A21) (thus obtaining T, cf (2.10)). Then substitute from (A27) and (A24) into (A1) (thus obtaining S, cf (2.11)).

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